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# Exact and asymptotic degeneracies of small black holes

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Abstract: We examine the recently proposed relations between black hole entropy and the topological string in the context of type II/heterotic string dual models. We consider the degeneracies of perturbative heterotic BPS states. In several examples with $\mathcal{N} = 4$ and $\mathcal{N} = 2$ supersymmetry, we show that the macroscopic degeneracy of small black holes agrees to all orders with the microscopic degeneracy, but misses non-perturbative corrections which are computable in the heterotic dual. Using these examples we refine the previous proposals and comment on their domain of validity as well as on the relevance of helicity supertraces.

Keywords: Superstrings and Heterotic Strings, Topological Strings, Black Holes in String Theory, String Duality.

## Contents





## 1. Introduction

One of the distinct successes of string theory is that, in some examples, it gives an account of black hole entropy in terms of statistical counting of microstates [1, 2]. One particularly rich set of examples are the BPS black holes associated with D-branes wrapped on Calabi-Yau manifolds in the type II string. In this case, the black hole solutions exhibit fixed-point attractor behavior near the horizon [3, 4]. Lopes Cardoso, de Wit, and Mohaupt [5 – 9] derived the generalized attractor equations in the presence of higher derivative F-type terms and obtained a formula for the Bekenstein-Hawking-Wald entropy of a black hole [10 – 14]. Recently, Ooguri, Strominger, and Vafa proposed that the thermodynamical ensemble implicit in the above entropy is a "mixed" ensemble where magnetic charges are treated micro-canonically while the electric ones are treated canonically [15]. This implies the following very elegant relation between the topological string associated to the Calabi-Yau manifold $\mathcal{X}$ [16 − 20] and the exact degeneracies of BPS states in the theory.

In the Type-IIA string, the relevant BPS states arise from wrapping D-branes on the various even cycles of the Calabi-Yau and hence carry electric and magnetic charges denoted by a vector $\gamma \in H^{\text{even}}(\mathcal{X}, \mathbf{Z})$. Upon choosing a symplectic splitting, one can define the (magnetic, electric) charge components of $\gamma$ as $(p^I, q_I), I = 0, 1, \ldots, h^{1,1}(\mathcal{X})$. Moreover, on the moduli space of complexified Kähler structures on $\mathcal{X}$, one has a set of "special coordinates" $\{X^I\}$. Let $F_{\text{top}}$ denote the holomorphic topological string partition function in these coordinates and define $\psi_p(\phi) = e^{F_{\text{top}}(p+i\phi)}$. Then, [15] proposes

$$\Omega(p, q) = \int d\phi \; |\psi_p|^2 e^{\pi q \cdot \phi} \,, \tag{1.1}$$

where $\Omega(p, q)$ denotes the number or perhaps the "index" of BPS states of charges $(p^I, q_I)$. A weaker form of the conjecture requires that this equation holds only to all orders in an asymptotic expansion in inverse charges [15]. Equation (1.1) has in turn been reformulated in terms of a pure density matrix in the geometric quantization of $H^3(\tilde{\mathcal{X}}, \mathbf{R})$ of the mirror Calabi-Yau $\tilde{\mathcal{X}}$ in the Type-IIB description [21].

While elegant, these formulae are somewhat imprecise. The measure $d\phi$ and the contour of integration in the integral have not been clearly specified, and the precise choice of definition of the microcanonical degeneracies $\Omega(p, q)$ has remained an issue. In this note we report on some attempts to refine the proposal (1.1), and to test its accuracy in explicit examples. A second paper in preparation will give further details [22].

In [23] it was pointed out that type IIA/heterotic duality offers a useful way to test (1.1), and this test was initiated for the standard example of the $\mathcal{N} = 4$ duality between the heterotic string on $T^6$ and the type IIA string on $K3 \times T^2$. The main point is that there is an interesting class of BPS states, the perturbative heterotic BPS states, (also known as Dabholkar-Harvey states, or DH states, for short [24, 25]), for which the exact degeneracies are known or can be deduced using available string technology. Moreover, much is known about the topological string partition function in these examples. The present paper develops further the use of type II/heterotic duality as a testing ground for (1.1).

The black holes corresponding to the DH states are mildly singular in the leading supergravity approximation. The geometry has a null singularity that coincides with the horizon and hence the classical area of these black holes vanishes [26, 27]. Effects of higher derivative terms in the string effective action are expected to modify the geometry [28, 29]. Indeed, for a subclass of higher derivative terms that are determined by the topological string amplitudes, the corrected black hole solution can be determined using the generalized attractor equations [5 − 8, 30]. The corrected solution has a smooth horizon with string scale area in the heterotic string metric [23, 31 − 34]. We refer to these black holes as 'small' black holes[1] to distinguish them from the 'large' black holes that have large classical area.

[1]The heterotic string coupling becomes very small at the horizon and as a result the horizon area is *large* in the duality invariant Einstein metric.



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Since small black holes have zero classical area, it is not *a priori* obvious that the formula (1.1) should apply. However, as noted above, the quantum corrected solution has a nonzero horizon area. Combined with the successful determination of degeneracies to all orders in $1/Q^2$ that we will find in §4, this gives strong evidence that there is nothing particularly pathological about these black holes. Nevertheless it should be borne in mind that the $\alpha'$ corrections to these geometries remain to be understood better.

We now give a brief overview of the remainder of the paper.

In §2, we show that in certain scaling limits of charges one can evaluate the integral (1.1) in a saddle point approximation that neglects the contributions of worldsheet instantons to $F_{\text{top}}$. We explain that this gives the leading asymptotic expansion to all orders in $1/Q^2$ where $Q$ is the graviphoton charge. We argue that the analysis can be reliably carried out for large black holes at both strong and weak coupling. Our analysis in fact suggests that the proposal (1.1) must be modified slightly. The modified version is given in eq. (2.31) below. As a matter of fact, one encounters serious difficulties in trying to make sense of the integral in (1.1) non-perturbatively. We comment on these difficulties, which arise mainly from the contribution of worldsheet instantons to the topological string amplitude, in §5.

In §3, we compute exactly the microscopic degeneracies of the DH states in a broad class of heterotic orbifolds with $\mathcal{N} = 4$ and $\mathcal{N} = 2$ supersymmetry and determine their asymptotics using the Rademacher formula reviewed in the Appendix. We also compute the "helicity supertraces" [35] that count the number of BPS short representations that cannot be combined into long representations. For $\mathcal{N} = 2$ compactifications this is the space-time counterpart of the "new supersymmetric index" on the worldsheet [36], as shown in [37–39]. One of the advantages of the states that we consider is that both the absolute number and the helicity supertraces are computable exactly.

In §4 we examine several $\mathcal{N} = 2$ and $\mathcal{N} = 4$ models in detail. In the $\mathcal{N} = 4$ examples we find remarkable agreement between the microscopic and macroscopic degeneracies to all orders in $1/Q^2$. This computation can be rigorously justified. In the $\mathcal{N} = 2$ examples of small black holes there turn out to be important subtleties in implementing the formalism of [15]. These are discussed in §2.4.4 and the conclusions.

In §5 we summarize our results, point out some open questions, and try to draw some lessons from what we have found.

Finally, we remark that there is a reciprocal version of the proposal [15]. In terms of this ensemble the formula of [5] is translated to:

$$e^{\mathcal{F}(p,\phi)} = \sum_q \Omega(p,q) e^{-\pi q \phi} \,. \tag{1.2}$$

Using our exact knowledge of degeneracies of DH states, one may try to construct the black hole partition function on the right-hand side and compare to the topological string amplitude. As we shall discuss in detail in [22], we find that the result bears a close resemblance to *a sum over translates* of the topological string amplitude, enforcing the expected periodicity under imaginary shifts $\phi \to \phi + 2i\mathbf{Z}$. This indicates that a theta series based on the topological string amplitude may be the appropriate monodromy-invariant object to represent the complete black-hole partition function [40].



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## 2. Macroscopic degeneracies via Saddle point approximation

### 2.1 Large radius limit

To determine the macroscopic degeneracies of small black holes, let us begin by attempting to evaluate the integral in (1.1) for a general compact Calabi-Yau manifold $\mathcal{X}$. The interpretation of $\psi_p$ as a wavefunction certainly suggests that (1.1) should be an integral over a vector space, and we expect it to be an integral over a real subspace of $H^{\text{even}}(\mathcal{X}, \mathbf{C})$. We will find below that the definition of the measure $d\phi$ is nontrivial, but for the moment we take it to be the standard euclidean measure.

Now, the holomorphic topological string partition function is only defined as an asymptotic expansion in the topological string coupling near some large radius limit (i.e. in a neighborhood of a point of maximal unipotent monodromy). In this limit we can write the holomorphic prepotential as a perturbative part plus a part due to worldsheet instantons. See for example [41, 42]. We will write

$$F_{\text{sugra}} = F^{\text{pert}} - \frac{iW^2}{2^7\pi} F^{GW} . \tag{2.1}$$

The perturbative part is

$$F^{\text{pert}} = -\frac{C_{abc}}{6}\frac{X^a X^b X^c}{X^0} - W^2 \frac{c_{2a}}{24 \cdot 64}\frac{X^a}{X^0} . \tag{2.2}$$

Here $a, b, c = 1, \ldots, h$, $h = h^{1,1}(\mathcal{X})$, label components with respect to an integral basis of $H_2(\mathcal{X}, \mathbf{Z})$ (which we also take to be a basis inside the Kähler cone), while $C_{abc}$ are the intersection numbers of dual 4-cycles of the Calabi-Yau. $c_{2a}$ are the components of the second Chern class. $W^2$ is the square of the Weyl superfield described in [9]. The sum over worldsheet instantons is

$$F^{GW} = \sum_{h \geq 0, \beta \in H_2(\mathcal{X}, \mathbf{Z})} N_{h,\beta}\, q^\beta \,\lambda^{2h-2} . \tag{2.3}$$

Here $N_{h,\beta}$ are the (rational) Gromov-Witten invariants,

$$q^\beta = e^{2\pi i \int_\beta (B+iJ)} = e^{2\pi i \beta_a \frac{X^a}{X^0}} \tag{2.4}$$

where $\beta_a \geq 0$ are components of $\beta$ with respect to an integral basis of $H_2(\mathcal{X}, \mathbf{Z})$, and $\lambda^2 = (\frac{\pi}{4X^0})^2 W^2$. In the topological string literature a slightly different normalization of the prepotential is used. The two are related by $F_{\text{sugra}} = -\frac{iW^2}{2^7\pi} F_{\text{top}}$. The attractor equations set $W^2 = 2^8$ so then $F_{\text{top}} = \frac{i\pi}{2} F_{\text{sugra}}$.

### 2.2 Perturbative evaluation

It is natural to expect that the "perturbative part" should give a good approximation to the integral, at least for large charges. We will discuss in detail what is meant by "large charges" in §2.4 below, where we will justify the procedure of looking for a consistent saddle point in (1.1) where it is a good approximation to replace $F_{\rm sugra}$ by $F^{\rm pert}$ defined in (2.2). Following [15] we must evaluate

$$\mathcal{F}^{\rm pert} := -\pi \ \operatorname{Im} F^{\rm pert}(p^I + i\phi^I, 256) \tag{2.5}$$

for $\phi^I$ real. We will set $p^0 = 0$ as this leads to significant simplifications. In this case we find that the perturbative part of the free energy is given by

$$\mathcal{F}^{\rm pert} = -\frac{\pi}{6}\frac{\hat{C}(p)}{\phi^0} + \frac{\pi}{2}\frac{C_{bc}(p)\phi^b\phi^c}{\phi^0} \tag{2.6}$$

where

$$C_{ab}(p) = C_{abc}p^c, \quad C(p) = C_{abc}p^a p^b p^c, \quad \hat{C}(p) = C(p) + c_{2a}p^a\,. \tag{2.7}$$

The perturbative part has a saddle point for

$$\phi^a_* = -C^{ab}(p)q_b\phi^0_*\,, \qquad \phi^0_* = \pm\sqrt{\frac{-\hat{C}(p)}{6\hat{q}_0}} \tag{2.8}$$

where $C^{ab}(p)$ is the inverse matrix of $C_{ab}(p)$ and

$$\hat{q}_0 = q_0 - \frac{1}{2}q_a C^{ab}(p) q_b \tag{2.9}$$

is the natural combinations of charges compatible with the unipotent monodromy. (In particular, $\hat{q}_0$ is monodromy invariant.)[2] In evaluating the saddle-point integral we must bear in mind that $C_{ab}(p)$ has indefinite signature (for example, for $p^a$ an ample divisor $C_{ab}(p)$ has signature $(1, h-1)$) and therefore $\phi^a\phi^b/\phi^0$ should be pure imaginary. We will take $p^a$ such that $\hat{C}(p) > 0$, and thus we want $\hat{q}_0 < 0$.

The integral (1.1), retaining only (2.6), is Gaussian on $\phi^a$ and of Bessel type for $\phi^0$. The precise choice of $\phi^0$ contour does not matter if we only concern ourselves with the asymptotic expansion of the $\phi^0$ integral for $\hat{C}(p)|\hat{q}_0| \to +\infty$. The asymptotics can then be given in terms of those of a Bessel function, the precise formula being:[3]

$$\mathcal{N}(p)\,\hat{I}_\nu\left(2\pi\sqrt{\frac{-\hat{C}(p)\hat{q}_0}{6}}\right) \tag{2.10}$$

where $\hat{I}_\nu(z)$ is related to the Bessel function $I_\nu(z)$ as in equation $(A.3)$ of the appendix and

$$\nu = \frac{1}{2}(n_v + 1)\,. \tag{2.11}$$

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[2]In general one should allow an extra quadratic polynomial in $X^I$ with real coefficients in $F^{\rm pert}$, say $-\frac{1}{2}A_{ab}X^aX^b - A_aX^aX^0 - A(X^0)^2$ where $A_{ab}, A_a, A$ are all *real*. The only effect of these terms in the present context is a shift of the charges to $\tilde{q}_a := q_a + A_{ab}p^b + A_ap^0, \tilde{q}_0 = q_0 + A_ap^a + 2Ap^0$. This will not affect our arguments so we drop these terms for simplicity.

[3]If we want to get the actual Bessel function from the $\phi^0$ integral then the appropriate contour to take is the circle described by $1/\phi^0 = -\epsilon + is$, $\epsilon > 0$, $s \in \mathbf{R}$. However, we should not discuss contours before the nonperturbative completion of $\psi_p$ is specified.



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Here $n_v$ is the rank of the total 4-dimensional gauge group, so $n_v = h + 1$. The Bessel function grows exponentially, for large Re$(z)$ (see $(A.6)$ ) so that the leading asymptotics of (2.10) agrees with the standard formula from [43] evaluated in the same limit. The factor $\mathcal{N}(p)$ is given by

$$\mathcal{N}(p) = \pm\frac{1}{2}\sqrt{\frac{1}{|\det C_{ab}(p)|}}\left(\frac{\hat{C}(p)}{6}\right)^{\nu} \tag{2.12}$$

and only depends on the magnetic charges $p^a$ and not on the electric charges $q_a$.

### 2.3 Modifications for small black holes

By definition, a small black hole is a BPS state such that $C(p) = 0$ but $\hat{C}(p) \neq 0$. In this case, while the horizon is singular and of zero area in the classical supergravity, it is expected that quantum corrections will smooth out the singularity leading to a legitimate black hole. For such charges, some of the manipulations in the previous section are not valid and must be modified as follows.

We are particularly interested in the case when $\mathcal{X}$ is a $K3$ fibration over $\mathbf{P}^1$ admitting a heterotic dual. Moreover, we are interested in charges corresponding, on the heterotic side, to DH states. As we will see, we cannot simply plug into (2.10). Nevertheless, a similar computation applies. If $\mathcal{X}$ is K3-fibered then we can divide up the special coordinates so that $X^1/X^0$ is the volume of the base and $X^a/X^0$, $a = 2, \ldots n_v - 1$ are associated with the (invariant part of the) Picard lattice of the fiber. The charges of heterotic DH states have $p^0 = 0$, $p^a = 0, a = 2, \ldots, h$, and $q_1 = 0$, with $p^1 q_0 \neq 0$ and $q_a \neq 0$ for $a = 2, \ldots, h$. In this case $C_{ab}(p)$ is of the form

$$p^1\begin{pmatrix} 0 & 0 \\ 0 & \tilde{C}_{a'b'} \end{pmatrix} \tag{2.13}$$

where $\tilde{C}_{a'b'}$ is the intersection form of the (invariant part of the) Picard lattice of the fiber. Note that now $C_{ab}(p)$ is not invertible. The $\phi^1$ dependence disappears from the integrand and one must make a discrete identification on $\theta = \phi^1/\phi^0$. One thereby finds that (1.1) gives

$$\mathcal{N}(p)\hat{I}_{\nu}\left(4\pi\sqrt{|p^1 q_0 - \frac{1}{2}q_{a'}\tilde{C}^{a'b'}q_{b'}|}\right) \tag{2.14}$$

where

$$\nu = \frac{1}{2}(n_v + 2) \tag{2.15}$$

and $\mathcal{N}(p)$ is a $p$-dependent prefactor.

Note that the argument of the Bessel function (2.10) nicely reduces to that of (2.14). For DH states, $C(p) = 0$, reflecting the fact that the classical area of the corresponding black holes is zero, and the nonzero entropy is provided by the quantum correction $c_{2a}p^a$. The change in index of the Bessel function results from an enhanced volume factor $\sqrt{\phi^0}$ arising from the zero mode of $C_{ab}(p)$.

Comparison with the exact results on DH degeneracies below shows that there is a non-trivial question of how to normalize the measure $d\phi$ (or the wavefunction $\psi_p$). In particular the $p$-dependent prefactors $\mathcal{N}(p)$ in (2.10) (2.14) are not compatible with exact results. An important point revealed by the case of small black holes is that the wavefunction $\psi_p$ is in fact not normalizable, at least, not in the conventional sense. We will return to this in the discussion section at the end.

## 2.4 Justification of the saddle point evaluation

### 2.4.1 Large charge limits

In this section we show that the perturbative evaluation of the integral performed above is valid provided we consider an appropriate scaling limit of large charges.

Let us begin by considering a rather general scaling limit of charges

$$\begin{aligned} \hat{q}_0 &\to s^x \hat{q}_0 \\ p^a &\to s^y p^a \end{aligned} \tag{2.16}$$

where $s \to \infty$. Here $x, y \geq 0$ and $p^a$ defines a vector in the Kähler cone. This scaling will result in a scaling

$$\phi^0_* \to s^z \phi^0_* + o(s^z) \tag{2.17}$$

for the saddle-point value $\phi^0_*$. Here $o(s^z)$ means terms growing strictly more slowly than $s^z$. (For example, from the saddle-point equation (2.20) below $z = (3y - x)/2$.) Now, there are three criteria we might wish to impose in order to be able to evaluate the integral (1.1) reliably in the saddle point approximation:

1. **Neglect of worldsheet instantons.** We expect the worldsheet instanton series to be small if $\mathrm{Im}\,\frac{X^a}{X^0} \gg 1$. In the saddle point approximation this means we require

$$-\frac{p^a}{\phi^0_*} \gg 1\,. \tag{2.18}$$

for all $a$. We fix the overall sign by choosing $p^a > 0$ and hence $\phi^0_* < 0$. Having all $p^a > 0$ means the divisor wrapped by the D4-brane is very ample. The above criterion requires $y > z$.

2. **Weak coupling in the expansion in $\lambda$.** A natural condition to require is that the topological string is weakly coupled. Physically, this is the requirement that the expansion of the supergravity effective action in powers of the graviphoton fieldstrength is not strongly coupled. Using the attractor value $W^2 = 2^8$ this means $\lambda = -4\pi i/\phi^0_*$ is small. Hence we require $z > 0$ for weak topological string coupling.

3. **Saddle-point equations.** We insist that $\phi^0_*$ satisfy the saddle point equations for the relevant approximation to $\mathcal{F}$. In the case of a weakly coupled topological string we must add the term

$$\Delta\mathcal{F} = \frac{\zeta(3)\chi(\mathcal{X})}{(4\pi)^2}(\phi^0)^2 := \frac{\pi}{2}\xi(\phi^0)^2 \tag{2.19}$$

to (2.6). Thus the explicit equations are

$$\begin{aligned} \frac{\hat{C}(p)}{6} + \hat{q}_0(\phi^0)^2 + \xi(\phi^0)^3 &= 0 \qquad \text{weak coupling} \\ \frac{\hat{C}(p)}{6} + \hat{q}_0(\phi^0)^2 &= 0 \qquad \text{strong coupling and/or} \quad \chi(\mathcal{X}) = 0\,. \end{aligned} \tag{2.20}$$

The full justification of the second line of (2.20) is given in §2.4.2.

There are two important subtleties in imposing the condition (2.18). First the Gromov-Witten series (2.3) includes the contribution of *pointlike instantons* with $\beta = 0$, and the criterion (2.18) does not lead to suppression of these terms, which must therefore be considered separately. Second there are further subtleties for small black holes discussed in 2.4.4 below. Sections 2.4.2 and 2.4.3 concern large black holes. Readers only interested in small black holes should skip to 2.4.4.

While weak coupling is a natural condition to impose, we will argue that it is not always necessary to do so, and of course one wants to understand both weak and strong coupling limits. In some cases, such as the small $\mathcal{N} = 4$ black holes, the computation of the macroscopic degeneracy can be fully justified at weak coupling (and turns out to be the same as at strong coupling).

### 2.4.2 Strong topological string coupling

There are certain charge limits of great interest in which one must work at strong topological string coupling. For example, in order to compare asymptotic degeneracies in the dual CFT description of [43] one requires that the level number be much larger than the central charge, and hence

$$|\hat{q}_0| \gg \hat{C}(p)\,. \tag{2.21}$$

(Validity of the supergravity approximation leads to a similar, but less restrictive criterion $|\hat{q}_0^3| \gg C(p)$ [43].) Equation (2.21) imposes the condition $x > 3y$ for large black holes. It is easy to see that in either case, the condition (2.21) is incompatible with (2.18) (2.20) and weak coupling. This motivates us to take a closer look at strong topological string coupling.

In this section we consider the limit of charges (2.21), and we will argue that it suffices to use the approximation (2.6) in this case. Thus, from (2.8) the topological string coupling $\lambda = -4\pi i/\phi^0_*$ is large, and therefore the topological string is strongly coupled.

In order to justify our procedure we separate the pointlike instantons from those with nonzero area by writing

$$F^{GW} = F^{GW}_{\beta=0} + F^{GW}_{\beta\neq 0}\,. \tag{2.22}$$

First, let us consider $F^{GW}_{\beta\neq 0}$. The worldsheet instanton corrections with $\beta \neq 0$ are formally suppressed by

$$\mathcal{O}\left(e^{-2\pi p^a \beta_a \sqrt{\frac{6|\hat{q}_0|}{\hat{C}(p)}}}\right) \tag{2.23}$$

where $\beta_a \geq 0$. Hence one may formally neglect the $\beta \neq 0$ terms in $F^{GW}$ up to exponentially small corrections. One should be careful at this point. Since the nonperturbative completion of the topological string is not known we must make an assumption. We will simply assume that $F^{GW}_{\beta\neq 0}$ has a nonperturbative completion so that the formal suppression (2.23) is valid, even though $\lambda \to \infty$. The justification of this assumption awaits a nonperturbative definition of the topological string. Nevertheless, let us note that this is a very reasonable assumption. The key point is that although the topological string coupling $\lambda$ goes to infinity, *the Kähler classes also go to infinity.*[4] The reason is that at the saddle point, $\operatorname{Im} t^a = |\lambda| p^a$. Thus the contribution $\lambda^{2h-2} q^\beta$ for $h > 1$ behaves like $\lambda^{2h-2} e^{-\kappa\lambda}$ where $\kappa$ is a positive constant. It therefore decays exponentially fast, even at strong coupling. More precisely, the contribution is

$$N_{h,\beta} \left( \frac{|\hat{q}_0|}{\hat{C}(p)} \right)^{h-1} e^{-2\pi p^a \beta_a \sqrt{\frac{6|\hat{q}_0|}{\hat{C}(p)}}} \tag{2.24}$$

and in the limit (2.21) this vanishes rapidly.

The above hypothesis can also be partially justified using the infinite product representation of $\exp F_{\mathrm{top}}$ implied by the work of Gopakumar and Vafa [44]. The infinite product may be split into three factors involving the BPS (a.k.a. Gopakumar-Vafa) invariants $n^{(h)}_\beta$ of spins $h = 0$, $h = 1$ and $h > 1$. The infinite products involving spin $h = 0$ and spin $h = 1$ BPS invariants can be shown to be convergent in appropriate domains, and they indeed satisfy our hypothesis. Unfortunately the infinite products involving spin $h > 1$ BPS invariants are in general not convergent. (The problem is that the maximal spin $h_*(\beta)$ for which $n^{(h)}_\beta$ is nonzero grows too rapidly with $\beta$.) Thus, in general, we cannot use the infinite product representation to give a nonperturbative definition. However, if $n^{(h)}_\beta = 0$ for $h > 1$ then our hypothesis is rigorously justified.

Now we must turn to the effects of the pointlike instantons contributing to $F^{GW}_{\beta=0}$. The results of [44] lead to a nonperturbative completion of $F^{GW}_{\beta=0}$. We have[5]

$$n^0_0 \left[ f(\lambda) + \frac{1}{12} \log \frac{\lambda}{2\pi i} - K \right] \sim \sum_h N_{h,0} \lambda^{2h-2} \tag{2.25}$$

where

$$\sum_h N_{h,0} \lambda^{2h-2} = -\frac{1}{2}\chi(\mathcal{X}) \left[ \lambda^{-2}\zeta(3) - \sum_{n=0}^{\infty} \lambda^{2n+2} \frac{|B_{2n+4}|}{(2n+4)!} \frac{(2n+3)}{(2n+2)} B_{2n+2} \right] \tag{2.26}$$

for $\lambda \to 0$. Here $n^0_0 = -\frac{1}{2}\chi(\mathcal{X})$, $K = -\frac{1}{24} - \frac{\zeta'(2)}{2\pi^2} + \frac{\gamma_E}{12}$ is a constant, and

$$f(\lambda) := \sum_{d=1}^{\infty} \frac{1}{d} \left( 2 \sin \frac{d\lambda}{2} \right)^{-2} = \log \prod_{k=1}^{\infty} \left( 1 - e^{i\lambda k} \right)^k \tag{2.27}$$

[4]This remark also resolves the following puzzle: If $\lambda$ is large one might expect the genus one term to dominate over the genus zero term. In fact, they are both of the same order, as is evident from (2.6).

[5]This identity is not stated correctly in the topological string theory literature, which omits the second and third terms on the left-hand side.



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(the second identity holds for $\mathrm{Im}\,\lambda > 0$). The important point is that the left-hand side of (2.27) is a well-defined function of $\lambda$, so long as $\lambda \notin \mathbf{R}$, and therefore defines a nonperturbative completion of $F^{\rm GW}_{\beta=0}$. Using the infinite-product (McMahon) formula for $f(\lambda)$ we have

$$e^{\mathcal{F}^{\rm GW}_{\beta=0}} = \left(-\frac{\phi^0}{2}\right)^{\chi/12} e^{K\chi} \left(\prod_{k\geq 1}(1-e^{\frac{4\pi k}{\phi^0}})^k\right)^{-\chi} \tag{2.28}$$

for $\phi^0 < 0$. Now, for $\phi^0 = -\sqrt{\hat{C}/6\hat{q}_0}$ negative and small, the infinite product is $1 + \mathcal{O}(e^{-4\pi\sqrt{6|\hat{q}_0|/\hat{C}}})$.

The factor $(-\frac{\phi^0}{2})^{\chi/12}$ in (2.28) will spoil the remarkable agreement between (1.1) and certain states in $\mathcal{N}=2$ models with $\chi \neq 0$, as described below. Therefore, to preserve this success we modify by hand the topological string wavefunction

$$\Psi_{top} \rightarrow \tilde{\Psi}_{top} := \lambda^{\chi/24} e^{F_{\rm top}} \tag{2.29}$$

so that

$$\tilde{\psi}_p(\phi) := \left(-\frac{\phi^0}{2}\right)^{-\chi/24} e^{F_{\rm top}(p+i\phi)} \tag{2.30}$$

and we propose a modification of the conjecture (1.1):

$$\Omega(p,q) = \mathcal{M}(p) \int d\phi |\tilde{\psi}_p(\phi)|^2 e^{\pi q\phi} \tag{2.31}$$

where $\mathcal{M}(p)$ depends on $p$ but not on $q$. This normalization factor is unavoidable; the $p$-dependent factor arising from the integrations, such as (2.12), in general does not agree with the $p$-dependent prefactor of the asymptotic expansion of the microscopic index.

To summarize, the integral in (2.31) may be defined as an asymptotic expansion in charges in the scaling limit (2.21). The value of the integral is

$$\mathcal{N}(p)\hat{I}_\nu\left(2\pi\sqrt{\frac{\hat{C}(p)|\hat{q}_0|}{6}}\right)\cdot\left(1+\mathcal{O}(e^{-\kappa(p)\sqrt{|\hat{q}_0|}})\right) \tag{2.32}$$

where $\mathcal{N}(p), \kappa(p)$ are $p$-dependent constants.

The modification (2.30) is very similar to an extra factor $\lambda^{\chi/24-1}$ which is included in the *nonholomorphic* topological string wavefunction. See [18, 19, 21]. We expect that taking proper account of measure factors in the definition of the wavefunction as a half-density will lead to a more satisfactory justification of our modification (2.29).

### 2.4.3 Weak topological string coupling

Now let us consider the situation for weak coupling. This can be achieved with a limit of charges with

$$y < x < 3y\,. \tag{2.33}$$



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If $\chi(\mathcal{X}) \neq 0$ then the saddle point equation in (2.20) has three roots. The discriminant is

$$\frac{\hat{C}}{12\xi}\left(\frac{\hat{C}}{12\xi}+2\left(\frac{\hat{q}_0}{3\xi}\right)^3\right)$$

and hence if $y < x$ there are three real roots of (2.20). One root $\phi^0_* \sim -\hat{q}_0/\xi + \cdots$ is inconsistent with large Kähler classes. The other two roots are

$$\phi^0_* = \pm\sqrt{\frac{\hat{C}(p)}{6|\hat{q}_0|}}\left(1 \mp \frac{1}{2}\xi\sqrt{\frac{\hat{C}(p)}{6|\hat{q}_0|^3}} + \cdots\right) \tag{2.34}$$

and as discussed earlier we choose the negative root. The saddlepoint evaluation of the integral is proportional to

$$(\det C_{ab}(p))^{-1/2}\int d\phi^0(\phi^0)^{h/2}\exp\left[-\frac{\pi\hat{C}}{6\phi^0}+\pi\hat{q}_0\phi^0+\frac{\pi}{2}\xi(\phi^0)^2+\sum_{h=2}^{\infty}N_{h,0}\left(\frac{4\pi i}{\phi^0}\right)^{2h-2}\right] \tag{2.35}$$

evaluated in an expansion around (2.34). (If we use the modified version (2.31) then we must replace $(\phi^0)^{h/2} \to (\phi^0)^{h/2-\chi/24}$ in (2.35).) The asymptotics will no longer be governed by a Bessel function, as in the strong couping regime. The leading correction to the entropy $2\pi\sqrt{\hat{C}|\hat{q}_0|/6}$ is no longer of order $\log s$, as in (2.10) but rather grows like a positive power of $s$:

$$S = 2\pi\sqrt{\frac{\hat{C}|\hat{q}_0|}{6}} + \frac{\zeta(3)\chi(\mathcal{X})}{96\pi^2}\frac{\hat{C}}{|\hat{q}_0|} + \cdots \tag{2.36}$$

It is an interesting challenge to reproduce this from a microscopic computation.[6] Finally, for completeness we note that if $x < y$ then (for $\chi \neq 0$) the roots are approximately $\phi^0 \sim (-\hat{C}/6\xi)^{1/3}$ and the Kahler classes are small. This means that in this regime of charges one must retain the full genus zero worldsheet instanton series.

#### 2.4.4 Additional subtleties for small black holes

In the case of small black holes $C(p) = 0$. Since the saddle point value of $\operatorname{Im} t^a = -p^a/\phi^0_*$, this implies that $C(\operatorname{Im} t) = 0$ and hence the saddle point is necessarily at the boundary of the Kähler cone. In principle, one must retain the full worldsheet instanton series (or rather, its analytic continuation, should that exist.) Remarkably, for $\mathcal{N} = 4$ compactifications this is not a problem. In this case $F_{\rm top}$ is only a function of a single Kähler modulus, namely, $t^1$ in the notation of §2.3. The reason is that the moduli space factors as a double-coset of $SL(2,\mathbf{R})$ times a Grassmannian, and by decoupling of vector and hypermultiplets, $F_{\rm top}$ must be constant on the Grassmannian factor. Moreover, in these compactifications $\chi(\mathcal{X}) = 0$ and hence the saddle-point values are:

$$\phi^0_* = -\sqrt{\frac{4p^1}{|\hat{q}_0|}} \qquad \operatorname{Im} t^1 = \frac{1}{2}\sqrt{p^1|\hat{q}_0|}\,. \tag{2.37}$$

Thus, whether or not the topological string coupling is strong ($|\hat{q}_0| \gg p^1$) or weak ($p^1 \gg |\hat{q}_0|$) the relevant Kähler class is large and the Bessel asymptotics (2.14) are justified.

[6] A similar correction has been computed in [5], without taking into account the contribution from the integration measure in (1.1).

The situation is rather different for $\mathcal{N} = 2$ compactifications. In this case $F_{top}$ is in general a function of $t^1$ as well as $t^a$ for $a \geq 2$. Thus the computation of section 2.3 is *not* justified. We stress that the problem is not that the topological string is strongly coupled. Indeed, for $\chi = 0$ examples such as the FHSV example discussed in §4.3 below, the saddlepoint value (2.37) can be taken in the weak coupling regime by taking $p^1 \gg |\hat{q}_0|$. In fact, the difficulty appears to be with the formulation of the integral (1.1) itself for the case of charges of small black holes. Recall that we must evaluate

$$\mathcal{F} := -\pi \ \operatorname{Im} F(p^I + i\phi^I, 256)\,. \tag{2.38}$$

Since $X^a/X^0 = \phi^a/\phi^0$ is *real*, for $a > 1$, one must evaluate the worldsheet instanton sum for real values $t^a = \phi^a/\phi^0$. For some Calabi-Yau manifolds it is possible to analytically continue the tree-level prepotential $F_0$ from large radius to small values of $\operatorname{Im} t^a$. However we may use the explicit results of [45, 46], which express $F_1 \sim \log \Phi$, where $\Phi$ is an automorphic form for $\mathrm{SO}(2, n; \mathbf{Z})$. It appears that $\operatorname{Im} t^a = 0$ consitutes a natural boundary of the automorphic form $\Phi$. Thus the formalism of [15] becomes singular for these charges, even at weak topological string coupling. Remarkably, if we ignore these subtleties, the formula (2.14) turns out to match perfectly with the asymptotic expansions of twisted sector DH states, as we show below. For untwisted sector DH states the asymptotics do not match with either the absolute degeneracies $\Omega_{abs}$ nor with the helicity supertrace $\Omega_2$ as we discuss in section 3.

### 2.5 Holomorphic vs. non-holomorphic topological string partition functions

The asymptotic expansion of the integral (2.31) differs from the entropy predicted from the attractor formalism, as modified in [5 − 9]. The latter identifies

$$S = \left[\mathcal{F} - \phi^I \frac{\partial \mathcal{F}}{\partial \phi^I}\right]_{\text{s.p.}}. \tag{2.39}$$

This is just the leading semiclassical approximation to (1.1) and does not capture the subleading corrections given by the asymptotics of the Bessel function. The same argument we have used to justify evaluating the integral (2.31) with $\mathcal{F}^{\text{pert}}$ can be applied to (2.39). After a suitable modification $\mathcal{F} \to \tilde{\mathcal{F}} = \mathcal{F} - \frac{\chi}{12} \log \phi^0$ the entropy given by (2.39) using the full nonperturbative prepotential $\tilde{\mathcal{F}}$ is the same as that given by $\mathcal{F}^{\text{pert}}$, up to exponentially small corrections. As we will see, this leads to predictions at variance with exact counting of heterotic BPS states.

Several recent papers [32, 47 − 49] have addressed this problem by taking into account the holomorphic anomaly in topological string theory. In particular, in the paper [48] the microscopic and macroscopic degeneracies for small black holes are shown to match in reduced rank $\mathcal{N} = 4$ models using a different ensemble than suggested by (1.1). Roughly speaking, the idea is that one has instead

$$S = \left[\mathcal{F}_{\text{eff}} - \phi^I \frac{\partial \mathcal{F}_{\text{eff}}}{\partial \phi^I}\right]_{\text{s.p.}}. \tag{2.40}$$

where $\mathcal{F}_{\text{eff}}$ is a non-Wilsonian, non-holomorphic effective action. On the other hand, it is clear from the discussion in [21] that one should use the *holomorphic* prepotential in (1.1) (2.31). These two approaches are not necessarily incompatible. The nonholomorphic effective action is obtained from the holomorphic Wilsonian effective action by integrating out massless modes. In a similar way $\mathcal{F}_{\text{eff}}$ might in fact be defined by carrying out the integral (1.1) (2.31).

## 3. Microscopic degeneracies of heterotic DH states

Let us now determine the microscopic degeneracies of the DH states using the heterotic dual. For concreteness, we will focus here on bosonic orbifolds of the heterotic string on $T^6$. (Using the elliptic genus it should be possible to extend the results in this section to a wider class of models.) We will denote the orbifold group by $\Gamma$. There is an embedding $R : \Gamma \to O(22) \times O(6)$. The orbifold group also acts by shifts so that the action on momentum vectors is

$$g|P\rangle = e^{2\pi i \delta(g)\cdot P}|R(g)P\rangle\,. \tag{3.1}$$

In $\mathbf{R}^{22,6}$, with metric $Diag(-1^{22}, +1^6)$ we can diagonalize the action of $R(g)$ with rotation angles $2\pi\theta_j(g)$, $j = 1, \ldots, 11$ on the leftmoving space and $2\pi\tilde{\theta}_j(g)$, $j = 1, 2, 3$ on the rightmoving space. The moduli are the boosts in $O(22, 6)$ commuting with the image $R(\Gamma)$. We consider embeddings $\Lambda \subset \mathbf{R}^{22,6}$ of $II^{22,6}$. We let $\Lambda(g)$ denote the sublattice of vectors fixed by the group element $g$. Of course, there will be constraints from level matching and anomaly cancellation. We assume that those constraints are satisfied. This still leaves a large class of possibilities.

$\mathcal{N} = 1$ spacetime supersymmetry requires that $\sum_i \tilde{\theta}_i(g) = 0$ mod 1 for all $g$. $\mathcal{N} = 2$ spacetime supersymmetry requires that $\tilde{\theta}_3(g) = 0$ for all $g$. In this case we let $\tilde{\theta}(g) := \tilde{\theta}_1(g) = -\tilde{\theta}_2(g)$. $\mathcal{N} = 4$ spacetime supersymmetry requires $\tilde{\theta}_i(g) = 0$ for all $i, g$.

The orbifold model will have a gauge symmetry. The currents in the Cartan subalgebra of the gauge symmetry (which is generically abelian) is spanned by $k$ pairs of left-moving bosons which are fixed for all $g \in \Gamma$, i.e. we suppose $\theta_i(g) = 0$ for all $g$ for $i = 1, \ldots, k$.[7] There is a subspace $\mathcal{Q} \subset \mathbf{R}^{22,6}$ fixed by all group elements. It is of signature $(2k, 6)$ for $\mathcal{N} = 4$ compactifications and $(2k, 2)$ for $\mathcal{N} = 2$ compactifications, respectively. The vectormultiplet moduli come from the SO$(2k, 6)$ (SO$(2k, 2)$ respectively) rotations in this plane. The number of $U(1)$ vector fields is $n_v = 2k + 6$ in the $\mathcal{N} = 4$ compactifications and $n_v = 2k+2$ in the $\mathcal{N} = 2$ compactifications. The lattice of electric charges (in the untwisted sector) for the gauge symmetry is the orthogonal projection (in the metric $(-1^{22}, +1^6)$) of $\Lambda$ into the plane $\mathcal{Q}$. Denote the charge lattice in the untwisted sector by $M_0$ and let $Q_{el} : \Lambda \to M_0$ be the orthogonal projection. States in the untwisted sector are naturally labelled by $P \in II^{22,6}$ but we only want to compute degeneracies at a fixed charge vector $Q \in M_0$.

[7]For brevity we restrict some generality. It is possible to have $\theta_i(g) = \frac{1}{2}$ allowing an odd number of twisted bosons. The formulae below are easily modified to accommodate this case.

Let us now compute the degeneracies of the DH states. In the untwisted sector DH states are all contained in the subspace of the 1-string Hilbert space of the form

$$\mathcal{H}_{osc,L} \otimes \mathcal{H}_{mom} \otimes \tilde{\mathcal{H}}_{gnd} \tag{3.2}$$

satisfying $L_0 = \tilde{L}_0$. Here the three factors are leftmoving oscillators, momentum eigenstates, and rightmoving groundstates. One important subtlety which arises for $\mathcal{N} = 2$ compactifications is that, in general, even in this subspace the DH states span a proper subspace. The projection to the BPS states depends only on the momentum $P$ of the state and implements the BPS condition $M^2 = (Q_{el})^2$. Let $\Pi_{bps}(P)$ be 1 if this condition is satisfied, and zero otherwise. For some vectors $P$ we have $\Pi_{bps}(P) = 1$ throughout the entire moduli space. However there can also be "chaotic BPS states" for which $\Pi_{bps}(P) = 0$ generically but, on a subspace of hypermultiplet moduli space, jumps to one [39]. The space of BPS states is graded by the electric charge lattice $M_{el}$ (in general $M_0$ is a proper sublattice) and we denote by $\mathcal{H}_{BPS}(Q)$ the subspace with charge $Q$. We will be interested in several measures of the degeneracies of states. The absolute number is $\Omega_{abs}(Q) := \dim \mathcal{H}_{BPS}(Q)$. Because of the chaotic BPS states this is not a constant function on moduli space. Examples show that a more appropriate quantity for comparing to (1.1) are the helicity supertraces. These are defined by[8]

$$\Omega_n(Q) := \frac{1}{2^n}\left(y\frac{\partial}{\partial y}\right)^n |_{y=+1} \mathrm{Tr}_{\mathcal{H}_{BPS}(Q)}(-1)^{2J_3} y^{2J_3} \tag{3.3}$$

where $J_3$ is a generator of the massive little group in 4 dimensions. For $\mathcal{N} = 2$ compactifications the first nonvanishing supertrace is $\Omega_2(Q)$ and this appears to be the correct quantity to use when comparing with the integral (1.1). Only BPS states contribute to $\Omega_2(Q)$. For $\mathcal{N} = 4$ compactifications the first nonvanishing supertrace is $\Omega_4(Q)$. This only receives contributions from $\frac{1}{2}$-BPS states. For $\Omega_6(Q)$ both $\frac{1}{2}$- and $\frac{1}{4}$-BPS states contribute. Examples suggest that $\Omega_4(Q)$ is the appropriate index to use for $\frac{1}{2}$-BPS states. Clearly, a different index must be chosen for 1/4-BPS states, if eq. (1.1) is to continue to hold for them as well. $\Omega_6(Q)$ is then the only candidate in this case.

The evaluation of the partition function in the BPS subspace of (3.2) is largely standard. Care must be exercised in the evaluation of the momentum sum since we are only interested in the degeneracies of the BPS states at a fixed $Q \in M_{el}$. In the untwisted sector we should write the momentum contribution as:

$$\sum_{P\in\Lambda(g)} q^{\frac{1}{2}P_L^2}\bar{q}^{\frac{1}{2}P_R^2} e^{2\pi i\delta(g)P}\Pi_{bps}(P) = \sum_{Q\in M_0} q^{\frac{1}{2}Q_L^2}\bar{q}^{\frac{1}{2}Q_R^2}\mathcal{F}_{g,Q}(q) \tag{3.4}$$

where

$$\mathcal{F}_{g,Q}(q) = \sum_{P\in\Lambda(g), Q_{el}(P)=Q} q^{\frac{1}{2}(P_L^2-Q_L^2)} e^{2\pi i\delta(g)P}\Pi_{bps}(P) \tag{3.5}$$

[8]These supertraces generalize the "index" "vectors minus hypers" used in [39]. See [50] appendix G for a nice discussion of helicity supertraces.

Note we have used the BPS condition $P_R^2 = Q_R^2$, and due to this condition we can write $P_L^2 - Q_L^2 = P^2 - Q^2$. The function (3.5) is actually very simple in many important cases. For example if $\Lambda(g) \subset M_0$, which is typical if the fixed space under the group element $g$ coincides with $\mathcal{Q}$ then we simply have $\mathcal{F}_{g,Q}(q) = e^{2\pi i \delta(g)\cdot Q}$. For this reason it is useful to distinguish between "*minimal twists*", which leave only the subspace $\mathcal{Q}$ invariant (i.e. $0 < \theta_j(g) < 1$ for $j > k$) and nonminimal twists. For nonminimal twists the kernel of $Q_{el}$ will be nontrivial and $\mathcal{F}_{g,Q}(q)$ will be a theta function.

Putting all this together the degeneracies of untwisted sector BPS states are given by

$$\Omega_n(Q) = e^{4\pi Q_R^2} \int d\tau_1 \; q^{\frac{1}{2}Q_L^2} \bar{q}^{\frac{1}{2}Q_R^2} \mathcal{Z}_n \tag{3.6}$$

where

$$\mathcal{Z}_n = \frac{1}{|\Gamma|} \sum_{g\in\Gamma} \frac{1}{\eta^{2+2k}} \left[ \prod_{j=1}^{11-k} (-2\sin\pi\theta_j(g)) \frac{\eta}{\vartheta\left[{\frac{1}{2} \atop \frac{1}{2}+\theta_j(g)}\right](|\tau)} \right] w_n(g) \mathcal{F}_{g,Q}(q) \tag{3.7}$$

and $w_n(g)$ is given by

$$w_n(g) = \begin{cases} 16\cos\pi\tilde{\theta}_1(g)\cos\pi\tilde{\theta}_2(g)\cos\pi\tilde{\theta}_3(g) & n = abs \\ 2(\sin\pi\tilde{\theta}(g))^2 & n = 2 \\ \frac{3}{2} & n = 4 \\ \frac{15}{8}(2 - E_2(\tau)) & n = 6 \end{cases} \tag{3.8}$$

The formula (3.7) is exact. Quite generally, the partition functions are negative weight modular forms and the degeneracies are given by their Fourier coefficients. There is a general formula — the Rademacher expansion — for the coefficients of such modular forms which is exact and yet summarizes beautifully the asymptotic behavior of these coefficients. It expresses these coefficients as an infinite sum of $I$-Bessel functions and thus is very well suited to comparison with the integral expression (2.14). The Rademacher expansion is summarized in the appendix. Using the Rademacher expansion, the leading asymptotics for the degeneracies of DH states from the minimal twists is ($n \neq 6$ here):

$$\frac{1}{4|\Gamma|} \sum_{g\in\Gamma,\text{minimal}}{}' w_n(g) h(g) \prod_{j=1}^{11-k} (-2\sin\pi\theta_j(g)) |\Delta_g|^{k+2} \hat{I}_{k+2}\left(4\pi\sqrt{|\Delta_g|\frac{1}{2}Q^2}\right) \tag{3.9}$$

where

$$h(g) = \begin{cases} (-1)^{(12-k)/2} \sin(2\pi\delta(g)Q + \pi\sum_j \theta_j(g)) & \text{k even} \\ (-1)^{(11-k)/2} \cos(2\pi\delta(g)Q + \pi\sum_j \theta_j(g)) & \text{k odd} \end{cases} \tag{3.10}$$

and

$$\Delta_g := -1 + \frac{1}{2}\sum_{j=1}^{11-k} \theta_j(g)(1-\theta_j(g)), \qquad 0 < \theta_j(g) < 1 \tag{3.11}$$

is the oscillator ground state energy in the sector twisted by $g$. The prime on the sum indicates we only get contributions from $g$ such that $\Delta_g < 0$. For nonminimal twists there will be similar contributions as described above. In particular the index on the Bessel function will be the same, but (3.11) receives an extra nonnegative contribution from the shift $\delta$, and the coefficient $|\Delta_g|^{k+2}$ is modified (and still positive). In some examples the leading asymptotics is provided by the minimal twists alone. It is interesting to compare this with the twisted sectors. Since the sector $(1, g)$ always mixes with $(g, 1)$ under modular transformation, and since the oscillator groundstate energy is $-1$ in the untwisted sector, it is clear that for charges $Q$ corresponding to states in the twisted sector the asymptotics will grow like

$$\hat{I}_{k+2}\left(4\pi\sqrt{\frac{1}{2}Q^2}\right). \tag{3.12}$$

This is true both for the absolute number of BPS states and for the supertraces. Recall that $k+2 = \frac{1}{2}(n_v+2)$ for $\mathcal{N}=2$ compactifications, so we have agreement with (2.15).

There are some interesting general lessons we can draw from our result (3.9). Due to the factor $h(g)$ it is possible that the leading $I$-Bessel functions cancel for certain directions of $Q$. Moreover, a general feature of $\mathcal{N}=2$ compactifications is that $g=1$ does not contribute to $\Omega_2$ in (3.9). Then, since $|\Delta_g| < 1$ the degeneracies are exponentially smaller in the untwisted sector compared to those of the twisted sector. We will see an explicit example of this below. In contrast, for $\mathcal{N}=4$ compactifications, the $g=1$ term *does* contribute to $\Omega_4$, which thus has the same growth as in the twisted sector.

One general lesson seems to be that the degeneracies, and even their leading asymptotics can be sensitive functions of the "direction" of $Q$ in charge space. In general it is quite possible that the exact BPS degeneracies and their asymptotics will be subtle arithmetic functions of the charge vector $Q$.[9] In the physics literature it is taken for granted that there is a smooth function $S_n : H^{\mathrm{even}}(\mathcal{X}, \mathbf{R}) \to \mathbf{R}$ so that $S_n(sQ) \sim \log\Omega_n(sQ)$ for $s \to \infty$, but the true situation might actually be much more subtle. The Rademacher expansion shows that Fourier coefficients of negative weight modular forms have well-defined asymptotics governed by Bessel functions. By contrast, the Fourier coefficients $a_n$ of cusp forms of positive weight $w$ have a lot of "scatter" and can only be described by a probability distribution for $a_n/n^{(w-1)/2}$. (See e.g. [52] for an introduction to this subject.) It would be very interesting to know where the functions $\Omega_n(Q)$ fit in this dichotomy.

## 4. Examples

We now give some examples of the results one finds using these general techniques. More details can be found in [22].

### 4.1 $K3 \times T^2$

This is dual to the heterotic string on $T^6$. We have $\dim_{\mathcal{H}_{BPS}(Q)} = p_{24}(N)$ where $N-1 = \frac{1}{2}Q^2$ and $\eta^{-24} = q^{-1}\sum_{N=0}^{\infty} p_{24}(N)q^N$. The Rademacher expansion (equation ($A$.4) below)

[9] Such a phenomenon was conjectured based on other considerations in [51].

becomes

$$\dim \mathcal{H}_{BPS}(Q) = 16 \cdot \left[ \hat{I}_{13}\left(4\pi\sqrt{\frac{1}{2}Q^2}\right) - 2^{-14} e^{i\pi\frac{1}{2}Q^2} \hat{I}_{13}\left(2\pi\sqrt{\frac{1}{2}Q^2}\right) + \cdots \right]. \qquad (4.1)$$

For $\Omega_4$ we simply replace 16 by $\frac{3}{2}$. For $\Omega_6$ we find

$$\Omega_6(Q) = \frac{15}{8}\left(2+\frac{1}{2}Q^2\right)\hat{I}_{13}\left(4\pi\sqrt{\frac{1}{2}Q^2}\right) + \cdots \qquad (4.2)$$

and thus we conclude that the correct supertrace to use in (1.1) is $\Omega_4$, at least in this example. We thus see that - with a proper normalization of the measure $d\phi$ - the integral expression (1.1) agrees with the exact degeneracies to *all orders in* $1/Q^2$ *in the leading exponential.* We stress that this agreement arises just from using the perturbative piece of $F(X^I, W^2)$. This is essentially the result of [23]. We also note that a naive inclusion of the worldsheet instanton corrections does not lead to the subleading Bessel functions given by the Rademacher expansion.

### 4.2 A reduced rank $\mathcal{N} = 4$ model

Besides the simplest $K_3 \times T^2$ compactification, it is also possible to construct a large number of $\mathcal{N} = 4$ type II models by considering quotients of $K_3 \times T^2$ by an Enriques automorphism of $K_3$ combined with a translation on $T^2$. We consider the simplest model with 14 $\mathcal{N} = 4$ vector multiplets, corresponding to an Enriques involution with 8 odd two-cycles. It is related by heterotic/type II duality [53] to the $\mathbf{Z_2}$ orbifold of the $E_8 \times E_8$ string, where the $\mathbf{Z_2}$ action interchanges the two $E_8$ factors and simultaneously shifts half-way along a circle so that the twisted states are massive [54, 55]. The topological amplitude $F_1$ for this model has been computed in [56].

To apply the formalism of §3, consider vectors $(P_1, P_2, P_3, P_4)$ in $E_8(-1) \oplus E_8(-1) \oplus II^{1,1} \oplus II^{5,5}$ with orbifold action[10]

$$g|P_1, P_2, P_3, P_4\rangle = e^{2\pi i \delta \cdot P_3}|P_2, P_1, P_3, P_4\rangle \qquad (4.3)$$

where $2\delta \in II^{1,1}$ and $\delta^2 = 0$. The charge lattice is $M_{el} = M_0 + M_1$ where $M_0$ are the charges of the untwisted sector with

$$M_0 = E_8\left(-\frac{1}{2}\right) \oplus II^{1,1} \oplus II^{5,5} \qquad (4.4)$$

while

$$M_1 = E_8\left(-\frac{1}{2}\right) \oplus (II^{1,1} + \delta) \oplus II^{5,5} \qquad (4.5)$$

are the charges in the twisted sector. For charges in the untwisted sector we denote $Q = \left(\frac{1}{\sqrt{2}}(2P + \wp), P_3, P_4\right)$ where $P \in E_8(+1)$, and $\wp$ runs over a set of lifts of $E_8/2E_8$ to $E_8$. The absolute number of BPS states is given by

$$\dim \mathcal{H}_{BPS}(Q) = d_Q^u(N) \qquad (4.6)$$

[10]The notation $E_8(a)$ used here and below means that the $E_8$ lattice norm is scaled by an overall factor of $a$.

for $N+\Delta_\wp = \frac{1}{2}Q^2$ where

$$8\Theta_{E_8(2),\wp}(\tau)\frac{1}{\eta^{24}} + 8\delta_{\wp,0}e^{2\pi i\delta\cdot P_3}\frac{2^4}{\eta^{12}\vartheta_2^4} := q^{\Delta_\wp}\sum_{N=0}^{\infty} d_Q^u(N)q^N \tag{4.7}$$

with

$$\Theta_{E_8(2),\wp}(\tau) := \sum_{Q\in E_8(+1)} e^{2\pi i\tau\left(Q-\frac{1}{2}\wp\right)^2} \tag{4.8}$$

The second supertrace vanishes, while for $\Omega_4$ we should multiply by 3/32. This expression only depends on $\wp$ up to the action of the Weyl group of $E_8$. There are three orbits, of length $1, 120$ and 135 corresponding to the trivial, adjoint, and 3875 representations. For each of these (4.8) may be expressed in terms of theta functions. For the twisted sector we define

$$\frac{1}{2}\left(\frac{1}{\eta^{12}\vartheta_4^4} \pm \frac{1}{\eta^{12}\vartheta_3^4}\right) = q^{\Delta_\pm}\sum_{N\geq 0} d_\pm^t(N)q^N \tag{4.9}$$

with $\Delta_+ = -\dfrac{1}{2}, \Delta_- = 0$. The absolute number of twisted sector BPS states is given by

$$\dim \mathcal{H}_{BPS}(Q) = 16\begin{cases} d_+^t(N) & e^{i\pi Q^2} = -1 \\ d_-^t(N) & e^{i\pi Q^2} = +1 \end{cases} \tag{4.10}$$

where $N+\Delta_\pm = \frac{1}{2}Q^2$.

Applying the Rademacher expansion we find for $\Omega_{abs}(Q) = \dim\mathcal{H}_{BPS}(Q)$:

$$\begin{cases} \frac{1}{2}\hat{I}_9\left(4\pi\sqrt{\frac{1}{2}Q^2}\right) + 2^{-6}(15+16e^{2\pi i P\cdot\delta})\hat{I}_9\left(4\pi\sqrt{\frac{1}{4}Q^2}\right) + \cdots & |\mathcal{O}_\wp| = 1 \\ \frac{1}{2}\hat{I}_9\left(4\pi\sqrt{\frac{1}{2}Q^2}\right) + 2^{-6}\hat{I}_9\left(4\pi\sqrt{\frac{1}{4}Q^2}\right) + \cdots & |\mathcal{O}_\wp| = 120 \\ \frac{1}{2}\hat{I}_9\left(4\pi\sqrt{\frac{1}{2}Q^2}\right) - 2^{-6}\hat{I}_9\left(4\pi\sqrt{\frac{1}{4}Q^2}\right) + \cdots & |\mathcal{O}_\wp| = 135 \\ \frac{1}{2}\hat{I}_9\left(4\pi\sqrt{\frac{1}{2}Q^2}\right) - 2^{-6}e^{i\pi Q^2}\hat{I}_9\left(4\pi\sqrt{\frac{1}{4}Q^2}\right) + \cdots & Q\in M_1 \end{cases} \tag{4.11}$$

In the first three lines $Q\in M_0$ and $|\mathcal{O}_\wp|$ is the order of the $E_8$ Weyl group orbit of $\wp$. The leading term is independent of the orbit, and in rather nice agreement with (2.14).

### 4.3 The FHSV model

As our third example let us consider the FHSV model. This has $\mathcal{N}=2$ supersymmetry and is described in [57]. We denote momentum vectors by $(P_1, P_2, P_3, P_4)$ in $II^{9,1}\oplus II^{9,1}\oplus II^{1,1}\oplus II^{3,3}$ The $\mathbf{Z}_2$ acts as

$$|P_1,P_2,P_3,P_4\rangle \to e^{2\pi i\delta\cdot P_3}|P_2,P_1,P_3,-P_4\rangle \tag{4.12}$$

with $\delta$ the order two shift vector defined in [57] ($\delta^2 = \frac{1}{2}$). The $u(1)^{12}$ electric charge lattice is $M_{el} = M_0 + M_1$ where

$$M_0 = E_8\left(-\frac{1}{2}\right) \oplus II^{1,1}\left(\frac{1}{2}\right) \oplus II^{1,1} \tag{4.13}$$

$$M_1 = E_8\left(-\frac{1}{2}\right) \oplus II^{1,1}\left(\frac{1}{2}\right) \oplus (II^{1,1} + \delta) \tag{4.14}$$

States from the untwisted sector have charge vectors in $M_0$, while states from the twisted sector have charge vectors in $M_1$. In order to give the degeneracies of DH states we define

$$\begin{aligned}
\frac{2^6}{\eta^6\vartheta_2^6} &= q^{-1}\sum_{N=0}^{\infty} d^u(N)q^N \\
\frac{1}{2}\left(\frac{1}{\eta^6\vartheta_4^6} + \frac{1}{\eta^6\vartheta_3^6}\right) &= q^{-\frac{1}{4}}\sum_{N=0}^{\infty} d^t_+(N)q^N \\
\frac{1}{2}\left(\frac{1}{\eta^6\vartheta_4^6} - \frac{1}{\eta^6\vartheta_3^6}\right) &= q^{+\frac{1}{4}}\sum_{N=0}^{\infty} d^t_-(N)q^N \,.
\end{aligned} \tag{4.15}$$

Then, for the helicity supertrace in the untwisted sector we have the result:

$$\Omega_2(Q) = \begin{cases} e^{2\pi i Q\delta} d^u(N) & Q \in M_0' \\ 0 & Q \in M_0 - M_0' \end{cases} \tag{4.16}$$

where $N - 1 = \frac{1}{2}Q^2$ and $M_0'$ is the sublattice of vectors of the form $2P_1 \oplus 2P_2 \oplus P_3$ of $M_0$.

For the twisted sector, note that $Q \in M_1$ and hence $Q^2 \in \mathbf{Z} + \frac{1}{2}$ The exact second supertrace is

$$\Omega_2(Q) = \begin{cases} -16 d^t_+(N) & \text{for } e^{i\pi Q^2} = -i \\ -16 d^t_-(N) & \text{for } e^{i\pi Q^2} = +i \end{cases} \tag{4.17}$$

The oscillator level $N$ is related to the momentum by the condition $N + \Delta_\pm = \frac{1}{2}Q^2$ and the $\pm$ sign is correlated with the sign of (4.17). Note that the metric $II^{9,1}(\frac{1}{2}) \oplus (II^{1,1} + \delta)$ is used here.

Using the Rademacher expansion we have the asymptotics

$$\Omega_2(Q) = \begin{cases} 2^{-8} e^{2\pi i Q\cdot\delta}\left(1 - e^{i\pi Q^2/2}\right)\hat{I}_7\left(2\pi\sqrt{\frac{1}{2}Q^2}\right) + \mathcal{O}\left(e^{\pi\sqrt{Q^2/2}}\right) & Q \in M_0' \\ 0 & Q \in M_0 - M_0' \\ -2^{-3}\hat{I}_7\left(4\pi\sqrt{\frac{1}{2}Q^2}\right) + 2^{-11} i e^{i\pi Q^2}\hat{I}_7\left(2\pi\sqrt{\frac{1}{2}Q^2}\right) + \mathcal{O}\left(e^{\pi\sqrt{Q^2/2}}\right) & Q \in M_1 \end{cases} \tag{4.18}$$

Let us now compare these results with (2.14) (2.15) and hence with (1.1) (2.31). The degeneracies in the twisted sector are consistent with (2.14) but this does not appear to be the case for the untwisted sector, because the exponential growth is $\exp[2\pi\sqrt{\frac{1}{2}Q^2}]$.[11] It

[11]This discrepancy is avoided in a class of $\mathcal{N} = 2$ heterotic orbifolds where twisted states carry the same charges as untwisted states, hence dominate the helicity supertrace [22]. In the FHSV model, twisted and untwisted states can be distinguished by the moding of the winding number.

is interesting also to consider the absolute number of BPS states in the untwisted sector. These are given by $\dim \mathcal{H}_{BPS}(Q) = \alpha(N)$ where $N-1 = \frac{1}{2}Q^2$ and

$$\frac{8}{\eta^{24}}\mathcal{F}_{g,Q}(q) = q^{-1}\sum_{N\geq 0}\alpha(N)q^N . \tag{4.19}$$

For generic moduli, the asymptotics of the absolute number of BPS states is controlled by $\hat{I}_{13}(4\pi\sqrt{\frac{1}{2}Q^2})$. However $\mathcal{F}_{g,Q}$ is a function of moduli and on some subvarieties of moduli space $\mathcal{F}_{g,Q}$ can be enhanced to an $E_8$ theta function. In this case the absolute number of BPS states is enhanced to $\hat{I}_9(4\pi\sqrt{\frac{1}{2}Q^2})$. Thus, the leading exponential behavior is the desired $\exp[4\pi\sqrt{\frac{1}{2}Q^2}]$ but the logarithmic corrections are in fact moduli-dependent. This is to be contrasted with the supertrace $\Omega_2$, which is moduli independent, but for $Q \in M_0'$ goes like $\hat{I}_7(2\pi\sqrt{\frac{1}{2}Q^2})$, and is exponentially smaller than the absolute number of BPS states. Thus the exact degeneracies do not agree with (1.1) (2.31) with any natural interpretation of $\Omega$. However, as explained in §2.4.4 the integrals (1.1) (2.31) are highly singular. Thus the formalism of [15] breaks down and this discrepancy cannot be said to constitute a counterexample to the conjecture of [15].

### 4.4 Purely electric states

It is also instructive to consider purely electric states, i.e. those with $p^a = 0$ but $q_a \neq 0$. An interesting example where such states can be investigated in detail are the perturbative type II DH states in $K3\times T^2$ compactification. These states are obtained from fundamental type II strings with momentum and winding along the $T^2$ factor. These are purely electric states in the natural polarization for the type II string. They are related by $U$-duality to BPS states of $D2$ branes wrapping a $T^2$ and a holomorphic curve in the $K3$ surface. In this case $p^a = 0$, so the perturbative free energy (2.6) vanishes, while the complete free energy is given by

$$\mathcal{F}(\phi, p) = -\log|\Delta(\tau)|^2 \tag{4.20}$$

for $\tau = \phi^1/\phi^0$. As a consequence, the integral (1.1) is highly singular. Nevertheless we have (see [50], eqs. ($G$.24) and ($G$.25)):

$$\begin{aligned} \Omega_4(Q) &= 36\ \delta_{Q^2,0} \\ \Omega_6(Q) &= 90\ \delta_{Q^2,0} \end{aligned} \tag{4.21}$$

for charges $Q$ such as we have described. Meanwhile $\Omega_{abs}(Q)$ grows exponentially, like $\exp[2\pi\sqrt{Q^2}]$. Note that in contrast to the heterotic case, for $Q^2 \neq 0$ these states are $\frac{1}{4}$-BPS, despite the fact that their discriminant vanishes. Further discussion of these states, and related states in type $(4,0)/(2,2)$ duality pairs will be given in [22].

### 4.5 Large black holes and the $(0,4)$ CFT dual

Regrettably, there are no examples where the degeneracies of large black holes are known exactly. In principle the index $\Omega_2$ should be computable from a $(0,4)$ sigma model described in [43, 58], presumably from the elliptic genus of this model. While the sigma model is rather complicated, and has not been well investigated we should note that from the Rademacher expansion it is clear that the leading exponential asymptotics of negative weight modular forms depends on very little data. Essentially all that enters is the order of the pole and the negative modular weight. There are $c_L = C(p) + c_2 \cdot p = \hat{C}(p)$ real left-moving bosons. Since the sigma model is unitary, the relevant modular form has the expansion $q^{-c_L/24} + \cdots$. This gives the order of the pole, and thus we need only know the modular weight. This in turn depends on the number of left-moving noncompact bosons. Each noncompact boson contributes $w = -\frac{1}{2}$ to the modular weight. Now, the sigma model of [43] splits into a product of a relatively simple "universal factor" and a rather complicated "entropic factor," as described in [58]. Little is known about the entropic factor other than that it is a $(0,4)$ conformal theory with $c_R = 6k$, where $k = \frac{1}{6}C(p) + \frac{1}{12}c_2 \cdot p - 1$, where $p \in H^2(\mathcal{Z}, \mathbf{Z})$. The local geometry of the target space was worked out in [58]. Based on this picture we will assume the target space is compact and does not contribute to the modular weight. (Quite possibly the model is a "singular conformal field theory" in the sense of [59] because the surface in the linear system $|p|$ can degenerate along the discriminant locus. It is reasonable to model this degeneration using a Liouville theory, as in [59]. If this is the case we expect the entropic factor to contribute order one modular weight.) The universal factor is much more explicit. The target is $\mathbf{R}^3 \times S^1$, it has $(0,4)$ supersymmetry with $k = 1$ and there are $h - 1$ compact leftmoving bosons which are $N = 4$ singlets. They have momentum in the anti-selfdual part of $H^{1,1}(\mathcal{X}, \mathbf{Z})$ (anti-selfduality is defined by the surface in $|p|$). Since we fix these momenta we obtain $w = -\frac{1}{2}(h-1)$. Finally there are 3 noncompact left-moving bosons describing the center of mass of the black hole in $\mathbf{R}^3$. Thus, the net left-moving modular weight is $-(h+2)/2$. Now, applying the Rademacher expansion in the region $|\hat{q}_0| \gg \hat{C}(p)$ we find the elliptic genus is proportional to

$$\hat{I}_\nu\left(2\pi\sqrt{\frac{|\hat{q}_0|\hat{C}(p)}{6}}\right) \tag{4.22}$$

with $\nu = \frac{h+4}{2}$. This is remarkably close to (2.10)! Clearly, further work is needed here since it is likely there are a number of important subtleties in the entropic factor. Nevertheless, our argument suggests that a deeper investigation of the elliptic genus in this model will lead to an interesting test of (1.1) (or rather (2.31), since it must be done at strong topological string coupling) for the case of large black holes.

## 5. Conclusions

We have seen that the heterotic DH states and the corresponding small black holes provide a rich set of examples for testing the precise meaning and the range of validity of (1.1). We have computed exactly the absolute number of DH states in a large class of orbifold compactifications with $\mathcal{N} = 4$ and $\mathcal{N} = 2$ supersymmetry. We have also evaluated various supertraces which effectively count the number of 'unpaired' BPS short multiplets that do not have the spin content to combine into long multiplets. These supertraces provide valuable information about how the BPS spectrum is organized and are important for finding the correct interpretation of our results. Using these data, a far more detailed comparison of microscopic and macroscopic degeneracies can be carried out than is possible for large black holes. We summarize below our results along with a number of puzzles and open problems and conclude with possible interpretations.

### 5.1 Results

On the macroscopic side, the asymptotic black hole degeneracies are proportional to a Bessel function (2.10) (2.14). For heterotic DH states with a charge vector $Q$, the Bessel function is of the form $\hat{I}_\nu(4\pi\sqrt{Q^2/2})$ where the index $\nu$ is given in terms of the number of massless vector fields by (2.15). If instead one considers a limit of charges with weak topological string coupling and $\chi(\mathcal{X}) \neq 0$ then the asymptotics are far more more complicated than those of a Bessel function, and are given by (2.36), in leading order.

On the microscopic side, the absolute number of the untwisted DH states is given by the general formulae (3.6), (3.7). The asymptotic microscopic degeneracies of the untwisted states are given by (3.9) and of the twisted states by (3.12). These are both expressed in terms of an $I$-Bessel function. Asymptotically, the relevant supertraces are also Bessel functions. All these Bessel functions in general have different arguments and indices. Comparison of these asymptotic degeneracies reveals the following broad patterns which we have checked in a few explicit examples here and many other examples that will be reported in [22].

- In all reduced rank CHL-type orbifolds with $\mathcal{N} = 4$ supersymmetry, there is remarkable agreement between the microscopic and macroscopic degeneracies for all possible charge vectors in both twisted and untwisted sectors. See for example (4.11). The agreement holds to all orders in an asymptotic expansion in $1/Q^2$, but fails nonperturbatively.[12] It is noteworthy that this agreement uses only the perturbative part of the topological string partition function and worldsheet instantons play no role. The relevant helicity supertrace in this case is $\Omega_4$ which turns out to be proportional to the absolute number because the left-moving oscillators of the heterotic string do not carry any spacetime fermion numbers, so there are no intermediate BPS representations.

- In orbifolds with $\mathcal{N} = 2$ supersymmetry, the leading order microscopic entropy is determined entirely by the argument of the Bessel function and in all models it goes as $4\pi\sqrt{Q^2/2}$. This is expected from a general argument in [29] that if the entropies match in the toroidally compactified heterotic string, as they do [23], then they must also match in all $\mathcal{N} = 2$ orbifolds. The subleading terms however depend also on the index of the Bessel function and these match only for twisted states but not for the untwisted states. The relevant nonvanishing helicity supertrace in this case is $\Omega_2$.

[12]Nonperturbative discrepancies in the formula (1.1) have previously been addressed in [60, 61]. The systems discussed in these papers are very different from the compact Calabi-Yau case discussed in this paper.

For the twisted states, $\Omega_2$ is proportional to the absolute number. For the untwisted states, $\Omega_2$ is exponentially smaller than the absolute number because the argument of the corresponding Bessel function turns out to be $2\pi\sqrt{Q^2/2}$ and moreover the index is also different.

Unfortunately, as we have explained in section 2.4.4 in this case we cannot reliably compute the macroscopic degeneracy because the prescription in [15] forces us to work on the boundary of Teichmüller space, and $F_{\rm top}$ is singular on this locus. Nevertheless, remarkably, if we ignore this subtlety and consider the result (2.14) we find precise agreement for the twisted sector DH states. We find disagreement both with $\Omega_{abs}$ and with $\Omega_2$ for the untwisted sector DH states.

- We have focused in this paper on the heterotic DH states, but it is instructive to consider also the Type-II DH states, as discussed in section 4.4. In this case, since $p^I = 0$, the graviphoton charge vanishes and the integral (1.1) becomes quite singular, even in cases where the exact $F_{\rm top}$ is known. Moreover, even after the inclusion of the F-type terms, the geometry continues to have a null singularity and does not develop a regular horizon. It is not clear in this case how to apply the formalism implicit in (1.1) and it is likely that the D-type terms are important for desingularizing these solutions. These states will be discussed in more detail in [22].

### 5.2 Puzzles and open problems

Our results raise a number of questions and puzzles. Their resolution is essential for a correct interpretation of (1.1).

- An important assumption underlying (1.1) both for the large and small black holes is that the D-type terms in the low energy effective action do not contribute to the black hole entropy. A priori, it is far from clear if that is the case. The strikingly successful agreement for the large class of heterotic DH states in $\mathcal{N} = 4$ orbifolds strongly suggests that at least for this class of small black holes, the D-terms in fact do not modify the entropy. It is highly unlikely that various precise numerical factors could have come out right only accidentally. It is quite conceivable for instance that once the F-type quantum corrections generate a solution with a regular horizon, then on that background solution, the corrections from the D-type terms do not change the Wald entropy possibly because of the index structure of the background Riemann tensor and gauge fields. There are analogous situations where a similar phenomenon occurs, for example, in $AdS_5 \times S^5$ or in chiral null models, where the higher curvature terms do not alter the solution because of the specific details of the index structure. It would be very interesting to see explicitly if this is indeed the case for our small black holes. The Type-II DH states noted in the previous subsection also suggest that in general, the D-type terms *will* be important. In this case, the F-type terms are inadequate to desingularize the solution. Following the heuristic picture of the stretched horizon suggested in [28], one is then forced to include the D-type terms to obtain a solution with a regular horizon to be able to make a meaningful comparison with the microstates. This suggests that even for large black holes, whether or not the effect of D-type terms needs to be included may depend on the details of the model and on the class of states.

- We have seen that even in the successful cases, (1.1) (or rather, the more accurate (2.31)) is only true in perturbation theory. If one wishes to go beyond the asymptotic expansion and understand (1.1) as a statement about exact BPS degeneracies, then one must specify a nonperturbative definition of $\psi_p$ and must then specify carefully the region of integration. Regarding the first problem, the $K3 \times T^2$ example is of fundamental importance because the $K3 \times T^2$ wavefunction is known exactly. In this case we can say definitively that $\psi_p$ is not a normalizable wavefunction and therefore not in the Hilbert space [22]. It is important and interesting to investigate this issue for other Calabi-Yau manifolds, but without a nonperturbative definition it is impossible to make definitive statements. Nevertheless, in the examples of $\mathcal{X}$ with heterotic duals, the functions $F_g$ are automorphic functions of the $t^a$. See, for examples, [39, 62 − 67]. This is already sufficient knowledge to address to some extent the question of what contour of integration should be chosen for the $\phi^I$. We have seen that if we keep just the perturbative part of $F$ then it is natural to integrate $\phi^I$ along the imaginary axis. However, this is problematic if we wish to retain the worldsheet instanton corrections. When $t^a := X^a/X^0$ has a positive imaginary part the instanton series in (2.1) at fixed $g$, but summed over $\beta$ converges. Automorphic forms are highly singular when evaluated for $t^a$ purely real. This can already be seen in the $K3 \times T^2$ example, where one is evaluating $\Delta(\tau)$ for real $\tau$. If one tries instead to expand the integrand of (1.1) using the expansion in Gromov-Witten invariants one finds an infinite series of order one terms leading to a nonsensical result. (In particular, the expansion in worldsheet instantons does not lead to the subleading exponentials in the Rademacher expansion.) How then are we to understand (1.1)? One possibility is that the full nonperturbative topological string partition function defines an $n$-form $\omega_p = d\phi e^{\mathcal{F}}$ with singularities on $H^{even}(\mathcal{X}, \mathbf{C})$ and that certain periods of this form give $\Omega(p, q)$. Then our procedure above could be a saddle point approximation to such a contour integral, and the Bessel functions (2.10) (2.14) represent the full asymptotic expansion multiplying the leading exponential. At least this interpretation is consistent with the data provided by perturbative heterotic states.

- An interesting question raised by the subleading Bessel functions in the Rademacher expansion is that of their physical meaning. The subleading corrections to $p_{24}(N)$ in the case of $K3 \times T^2$ are down by $exp[-4\pi \frac{c-1}{c}\sqrt{N}]$, $c = 2, 3, \ldots$, and since $\sqrt{N} \sim 1/g_s^2$ at the horizon this is suggestive of some novel nonperturbative effects.

### 5.3 Interpretations

One interpretation that has been suggested in [15] is that the quantity $\Omega$ appearing in (1.1) is not the absolute number of micro-states, but rather an index. It is natural to identify this proposed index with $\Omega_4$ (or $\Omega_6$) in $\mathcal{N} = 4$ theories and with $\Omega_2$ in $\mathcal{N} = 2$ theories. In all successful examples where the agreement works, this index always equals the absolute number and also the macroscopic black hole degeneracy. This seems to support the above interpretation. However, the interpretation in terms of an index seems problematic from the point of view of thermodynamics. The Bekenstein-Hawking-Wald entropy appears in the first law of thermodynamics which can be derived in the Lorentzian theory where there are no ambiguities about fermionic boundary conditions. As with any other thermodynamic system, one should identify this entropy with the logarithm of the absolute number of microstates by the Boltzmann relation and not with an index. Generically, the index will be much smaller than the absolute number because many states can cancel in pairs when counted in an index and thus cannot equal the thermodynamic entropy. This problem is even more acute for large black holes. In this case, the classical area is finite and any possible quantum corrections due to the F-type and D-type terms are subleading. On general grounds, it does not seem reasonable to identify this thermodynamic entropy with an index. Our results suggest a possible alternative interpretation that the macroscopic entropy should be compared with the absolute microscopic degeneracies, but that these degeneracies must be computed in an appropriate "nonperturbative" regime of moduli space. Indeed this is what one would expect from the Boltzmann relation in conventional statistical mechanics.[13] Note that even if the string coupling remains small at the horizon it does not mean that we are in a perturbative regime because the graviphoton charge of the state of interest has to be large enough so that a black hole is formed. Formation of a black hole is clearly a nonperturbative change in the perturbative flat spacetime geometry. This is analogous to a situation in QED where even if the fine structure constant $\alpha$ is small, the interactions of a particle with charge $Z$ cannot be computed in perturbation theory for sufficiently large $Z$ once $\alpha Z$ is of order one.

Therefore, for a correct comparison, we need to evaluate the microscopic degeneracies in the regions of the moduli space determined by the attractor geometry where a black hole has formed. We are instead computing the microscopic degeneracies in the perturbative regime using free string theory in flat spacetime. The two computations do not always have to agree even for BPS states in short multiplets because with the right spin content, many short multiplets can in principle combine into a long multiplet. The long multiplets are then not protected from renormalization. This suggests that the spectrum of BPS short multiplets would be robust against renormalization only when their absolute number equals an index and that index is itself constant. In this case, the short multiplets cannot turn into a long multiplet because they simply do not have the required spin content. This interpretation is indeed consistent with our results for all heterotic DH states. Whenever the perturbative microscopic degeneracies match with macroscopic degeneracies as in the $\mathcal{N}=4$ models or for the twisted states in the $\mathcal{N}=2$ models, they also equal an index. It seems reasonable to expect that in this case the microscopic degeneracies in the nonperturbative black hole regime can be reliably deduced from the microscopic degeneracies in the perturbative regime.

[13] In fact, not only $\Omega_{abs}$ but also $\Omega_2$ is only a locally constant function on moduli space. The function $\Omega_2$ can change across walls of marginal stability in vectormultiplet moduli space (although it is constant in hypermultiplet moduli space). Thus, even a version of (1.1) in which $\Omega$ is given by an index must also take into account the region of moduli space in which $\Omega$ is being computed.



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**Note added.** Both versions 1 and 2 of this paper asserted that the degeneracies of untwisted DH states in $\mathcal{N}=2$ orbifold compactifications constituted a counterexample to the conjecture of [15]. We subsequently realized that in these examples our computation of the integral (1.1) in §2.3 is not rigorous because certain Kähler classes are zero at the attractor point.[14] For further explanation and discussion see §2.4.4 and §5.1. In the present revised version, our claims are requalified as follows: we find rigorous agreement for $\mathcal{N}=4$ compactifications, remarkable *unjustified* agreement for twisted sector $\mathcal{N}=2$ DH states, and *apparent* discrepancy for untwisted $\mathcal{N}=2$ DH states. In fact, the formula (1.1) appears to be rather singular in this case. We have also taken the opportunity to add some new results in §2.4.3 and §4.5.

## A. The Rademacher expansion

Here we state briefly the Rademacher expansion. For more details and information see [69]. Suppose we have a "vector-valued nearly holomorphic modular form," i.e. , a collection of functions $f_\mu(\tau)$ which form a finite-dimensional unitary representation of the modular group of weight $w<0$. Under the standard generators we have

$$\begin{aligned} f_\mu(\tau+1) &= e^{2\pi i\Delta_\mu} f_\mu(\tau) \\ f_\mu(-1/\tau) &= (-i\tau)^w S_{\mu\nu} f_\nu(\tau) \end{aligned} \tag{A.1}$$

We assume the $f_\mu(\tau)$ have no singularities for $\tau$ in the upper half plane, except at the cusps $\mathbf{Q}\cup i\infty$. We may assume they have an absolutely convergent Fourier expansion

$$f_\mu(\tau) = q^{\Delta_\mu}\sum_{m\geq 0} F_\mu(m) q^m \qquad \mu=1,\ldots,r \tag{A.2}$$

with $F_\mu(0)\neq 0$ and that the $\Delta_\mu$ are real. We wish to give a formula for the Fourier coefficients $F_\mu(m)$. Define:

$$\hat{I}_\nu(z) = -i(2\pi)^\nu \int_{\epsilon-i\infty}^{\epsilon+i\infty} t^{-\nu-1} e^{(t+z^2/(4t))} dt = 2\pi\left(\frac{z}{4\pi}\right)^{-\nu} I_\nu(z) \tag{A.3}$$

for $Re(\nu)>0, \epsilon>0$, where $I_\nu(z)$ is the standard modified Bessel function of the first kind.

[14] We disagree with the statement in footnote 2 of [68]. In fact, for the FHSV example the computation can be done at weak coupling. In particular, the nonperturbative effects discussed in [68] , of order $\mathcal{O}(e^{-t^2/\lambda})$, are exponentially small in the limit we consider.

Then we have:

$$F_\nu(n) = \sum_{c=1}^{\infty} \sum_{\mu=1}^{r} c^{w-2} K\ell(n,\nu,m,\mu;c) \sum_{m+\Delta_\mu<0} F_\mu(m)$$
$$|m+\Delta_\mu|^{1-w} \qquad \hat{I}_{1-w}\left[\frac{4\pi}{c}\sqrt{|m+\Delta_\mu|(n+\Delta_\nu)}\right]. \tag{A.4}$$

The coefficients $K\ell(n,\nu,m,\mu;c)$ are generalized Kloosterman sums. For $c=1$ we have:

$$K\ell(n,\nu,m,\mu;c=1) = S^{-1}_{\nu\mu} \tag{A.5}$$

The series (A.4) is convergent. Moreover the asymptotics of $I_\nu$ for large $\mathrm{Re}(z)$ is given by

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}\left[1 - \frac{(\mu-1)}{8z} + \frac{(\mu-1)(\mu-3^2)}{2!(8z)^2} - \frac{(\mu-1)(\mu-3^2)(\mu-5^2)}{3!(8z)^3} + \dots\right], \tag{A.6}$$

where $\mu = 4\nu^2$.

## References


[1] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett.* **B 379** (1996) 99 [hep-th/9601029].

[2] T. Damour, *The entropy of black holes: a primer*, hep-th/0401160.

[3] S. Ferrara, R. Kallosh and A. Strominger, *$N=2$ extremal black holes*, *Phys. Rev.* **D 52** (1995) 5412 [hep-th/9508072].

[4] A. Strominger, *Macroscopic entropy of $N=2$ extremal black holes*, *Phys. Lett.* **B 383** (1996) 39 [hep-th/9602111].

[5] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Corrections to macroscopic supersymmetric black-hole entropy*, *Phys. Lett.* **B 451** (1999) 309 [hep-th/9812082].

[6] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Deviations from the area law for supersymmetric black holes*, *Fortschr. Phys.* **48** (2000) 49 [hep-th/9904005].

[7] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Area law corrections from state counting and supergravity*, *Class. and Quant. Grav.* **17** (2000) 1007 [hep-th/9910179].

[8] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Macroscopic entropy formulae and non-holomorphic corrections for supersymmetric black holes*, *Nucl. Phys.* **B 567** (2000) 87 [hep-th/9906094].

[9] T. Mohaupt, *Black hole entropy, special geometry and strings*, *Fortschr. Phys.* **49** (2001) 3 [hep-th/0007195].

[10] J.D. Bekenstein, *Black holes and entropy*, *Phys. Rev.* **D 7** (1973) 2333.

[11] J.D. Bekenstein, *Generalized second law of thermodynamics in black hole physics*, *Phys. Rev.* **D 9** (1974) 3292.

[12] S.W. Hawking, *Particle creation by black holes*, *Commun. Math. Phys.* **43** (1975) 199.

[13] R.M. Wald, *Black hole entropy in the noether charge*, *Phys. Rev.* **D 48** (1993) 3427 [gr-qc/9307038].

[14] V. Iyer and R.M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev.* **D 50** (1994) 846 [gr-qc/9403028].

[15] H. Ooguri, A. Strominger and C. Vafa, *Black hole attractors and the topological string*, *Phys. Rev.* **D 70** (2004) 106007 [hep-th/0405146].

[16] E. Witten, *Topological sigma models*, *Commun. Math. Phys.* **118** (1988) 411.

[17] E. Witten, *Mirror manifolds and topological field theory*, hep-th/9112056.

[18] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311 [hep-th/9309140].

[19] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Holomorphic anomalies in topological field theories*, *Nucl. Phys.* **B 405** (1993) 279 [hep-th/9302103].

[20] I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, *Topological amplitudes in string theory*, *Nucl. Phys.* **B 413** (1994) 162 [hep-th/9307158].

[21] E.P. Verlinde, *Attractors and the holomorphic anomaly*, hep-th/0412139.

[22] A. Dabholkar, F. Denef, G.W. Moore and B. Pioline, *Precision counting of small black holes*, hep-th/0507014.

[23] A. Dabholkar, *Exact counting of black hole microstates*, *Phys. Rev. Lett.* **94** (2005) 241301 [hep-th/0409148].

[24] A. Dabholkar and J.A. Harvey, *Nonrenormalization of the superstring tension*, *Phys. Rev. Lett.* **63** (1989) 478.

[25] A. Dabholkar, G.W. Gibbons, J.A. Harvey and F. Ruiz Ruiz, *Superstrings and solitons*, *Nucl. Phys.* **B 340** (1990) 33.

[26] A. Sen, *Black hole solutions in heterotic string theory on a torus*, *Nucl. Phys.* **B 440** (1995) 421 [hep-th/9411187].

[27] A. Dabholkar, J.P. Gauntlett, J.A. Harvey and D. Waldram, *Strings as solitons and black holes as strings*, *Nucl. Phys.* **B 474** (1996) 85 [hep-th/9511053].

[28] A. Sen, *Extremal black holes and elementary string states*, *Mod. Phys. Lett.* **A 10** (1995) 2081 [hep-th/9504147].

[29] A. Sen, *Black holes and elementary string states in $N = 2$ supersymmetric string theories*, *JHEP* **02** (1998) 011 [hep-th/9712150].

[30] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Stationary BPS solutions in $N = 2$ supergravity with $R^2$ interactions*, *JHEP* **12** (2000) 019 [hep-th/0009234].

[31] A. Dabholkar, R. Kallosh and A. Maloney, *A stringy cloak for a classical singularity*, *JHEP* **12** (2004) 059 [hep-th/0410076].

[32] A. Sen, *How does a fundamental string stretch its horizon?*, *JHEP* **05** (2005) 059 [hep-th/0411255].

[33] V. Hubeny, A. Maloney and M. Rangamani, *String-corrected black holes*, *JHEP* **05** (2005) 035 [hep-th/0411272].

[34] D. Bak, S. Kim and S.-J. Rey, *Exactly soluble BPS black holes in higher curvature $N = 2$ supergravity*, hep-th/0501014.




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[35] S. Ferrara, C.A. Savoy and L. Girardello, *Spin Sum Rules In Extended Supersymmetry*, *Phys. Lett.* **B 105** (1981) 363.

[36] S. Cecotti, P. Fendley, K.A. Intriligator and C. Vafa, *A new supersymmetric index*, *Nucl. Phys.* **B 386** (1992) 405 [hep-th/9204102].

[37] W. Lerche, *Elliptic index and superstring effective actions*, *Nucl. Phys.* **B 308** (1988) 102.

[38] W. Lerche, A.N. Schellekens and N.P. Warner, *Ghost triality and superstring partition functions*, *Phys. Lett.* **B 214** (1988) 41.

[39] J.A. Harvey and G.W. Moore, *Algebras, BPS states and strings*, *Nucl. Phys.* **B 463** (1996) 315 [hep-th/9510182].

[40] B. Pioline, *BPS black hole degeneracies and minimal automorphic representations*, hep-th/0506228.

[41] S. Hosono, A. Klemm and S. Theisen, *Lectures on mirror symmetry*, hep-th/9403096.

[42] S. Hosono, A. Klemm, S. Theisen and S.T. Yau, *Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces*, *Nucl. Phys.* **B 433** (1995) 501 [hep-th/9406055].

[43] J.M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M-theory*, *JHEP* **12** (1997) 002 [hep-th/9711053].

[44] R. Gopakumar and C. Vafa, *M-theory and topological strings. II*, hep-th/9812127.

[45] J.A. Harvey and G.W. Moore, *Exact gravitational threshold correction in the FHSV model*, *Phys. Rev.* **D 57** (1998) 2329 [hep-th/9611176].

[46] M. Henningson and G.W. Moore, *Threshold corrections in $K(3) \times T(2)$ heterotic string compactifications*, *Nucl. Phys.* **B 482** (1996) 187 [hep-th/9608145].

[47] G.L. Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Asymptotic degeneracy of dyonic $N = 4$ string states and black hole entropy*, *JHEP* **12** (2004) 075 [hep-th/0412287].

[48] A. Sen, *Black holes, elementary strings and holomorphic anomaly*, hep-th/0502126.

[49] B. de Wit, *Introduction to black hole entropy and supersymmetry*, hep-th/0503211.

[50] E. Kiritsis, *Introduction to superstring theory*, hep-th/9709062.

[51] S.D. Miller and G.W. Moore, *Landau-siegel zeroes and black hole entropy*, hep-th/9903267.

[52] S.S. Gelbart and S.D. Miller, *Riemann's zeta function and beyond*, *Bull. Am. Math. Soc.* **41** (2004) 59.

[53] J.H. Schwarz and A. Sen, *Type IIa dual of the six-dimensional CHL compactification*, *Phys. Lett.* **B 357** (1995) 323 [hep-th/9507027].

[54] S. Chaudhuri, G. Hockney and J.D. Lykken, *Maximally supersymmetric string theories in $D < 10$*, *Phys. Rev. Lett.* **75** (1995) 2264 [hep-th/9505054].

[55] S. Chaudhuri and J. Polchinski, *Moduli space of CHL strings*, *Phys. Rev.* **D 52** (1995) 7168 [hep-th/9506048].

[56] A. Gregori et al., *$R^2$ corrections and non-perturbative dualities of $N = 4$ string ground states*, *Nucl. Phys.* **B 510** (1998) 423 [hep-th/9708062].

[57] S. Ferrara, J.A. Harvey, A. Strominger and C. Vafa, *Second quantized mirror symmetry*, *Phys. Lett.* **B 361** (1995) 59 [hep-th/9505162].

[58] R. Minasian, G.W. Moore and D. Tsimpis, *Calabi-Yau black holes and (0,4) sigma models*, *Commun. Math. Phys.* **209** (2000) 325 [hep-th/9904217].

[59] N. Seiberg and E. Witten, *The D1/D5 system and singular CFT*, *JHEP* **04** (1999) 017 [hep-th/9903224].

[60] C. Vafa, *Two dimensional Yang-Mills, black holes and topological strings*, hep-th/0406058.

[61] M. Aganagic, H. Ooguri, N. Saulina and C. Vafa, *Black holes, q-deformed* 2*d Yang-Mills and non-perturbative topological strings*, *Nucl. Phys.* **B 715** (2005) 304 [hep-th/0411280].

[62] L.J. Dixon, V. Kaplunovsky and J. Louis,*Moduli dependence of string loop corrections to gauge coupling constants*, *Nucl. Phys.* **B 355** (1991) 649.

[63] B.de Wit, V. Kaplunovsky, J. Louis and D. Lüst, *Perturbative couplings of vector multiplets in $N = 2$ heterotic string vacua*, *Nucl. Phys.* **B 451** (1995) 53 [hep-th/9504006].

[64] I. Antoniadis, S. Ferrara, E. Gava, K.S. Narain and T.R. Taylor, *Perturbative prepotential and monodromies in $N = 2$ heterotic superstring*, *Nucl. Phys.* **B 447** (1995) 35 [hep-th/9504034].

[65] G.W. Moore, *String duality, automorphic forms and generalized Kac-Moody algebras*, *Nucl. Phys.* **67** *(Proc. Suppl.)* (1998) 56 [hep-th/9710198].

[66] M. Marino and G.W. Moore, *Counting higher genus curves in a Calabi-Yau manifold*, *Nucl. Phys.* **B 543** (1999) 592 [hep-th/9808131].

[67] A. Klemm, M. Kreuzer, E. Riegler and E. Scheidegger, *Topological string amplitudes, complete intersection Calabi-Yau spaces and threshold corrections*, *JHEP* **05** (2005) 023 [hep-th/0410018].

[68] R. Dijkgraaf, R. Gopakumar, H. Ooguri and C. Vafa, *Baby universes in string theory*, hep-th/0504221.

[69] R. Dijkgraaf, J.M. Maldacena, G.W. Moore and E.P. Verlinde, *A black hole Farey tail*, hep-th/0005003.


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